Int J Theor Phys (2010) 49: 1244–1250
DOI 10.1007/s10773-010-0306-5

Controlled Dense Coding with Five-Qubit Cluster State

Cui-Lan Luo · Guo-Qiang Huang

Received: 30 December 2009 / Accepted: 3 March 2010 / Published online: 19 March 2010

Abstract Two schemes, via entanglement concentration and with generalized measurement respectively, for controlled dense coding with a one-dimensional five-qubit cluster state are investigated. In this protocol, the supervisor (Cliff) can control the entanglement of the channel and the average amount of information transmitted from the sender (Alice) to the receiver (Bob) by adjusting the local measurement angle θ. It is shown that the results for the average amounts of information are unique from the different two schemes.

Keywords Controlled dense coding · Five-qubit cluster state · Average amount of information · POVM

1 Introduction

Quantum entanglement is a quintessential property of quantum mechanics that sets it apart from any classical physical theory. An important feature of entanglement is that it gives rise to correlations that cannot be explained by any local realistic description of quantum mechanics. In recent years, quantum entanglement has become an important physical resource for teleportation [1–6], dense coding [7, 8], quantum state sharing [9–13], and quantum computation [14]. Dense coding, or superdense coding, is one of the important branches of quantum information theory. In the original protocol, the authors have showed how the entangled states can increase the communication capacity of two parties. In an ideal classical channel, the transmission of 2 bits of information requires the manipulation and transmission of at least two particles or physical entities, which are used to encode the transmitted information. However, if the sender and the receiver share a maximally bipartite entangled state, the sender can transmit 2 bits of information by manipulating and sending only one qubit [7, 8].

C.-L. Luo (✉) · G.-Q. Huang
Jiangxi University of Finance and Economics, Nanchang 330013, China
e-mail: clluohuang@sina.com

G.-Q. Huang
e-mail: guoqhuang@tom.com

Since the original protocol of quantum dense coding considered in 1992 by Bennett and Wiesner, dense coding has been generalized in various directions. It has been generalize not only to the continuous variables [15, 16], but also to the multipartite communication [17–20]. One can perform dense coding not only with quantum states in finite dimensional Hilbert spaces but also with quantum states in infinite dimensional Hilbert spaces. Quantum dense coding was experimentally presented by Mattle et al. in an optical system [21], and then by Fang et al. with the nuclear magnetic resonance techniques [22].

The first controlled dense coding protocol was proposed by Hao et al. in 2001 [23], where the sender Alice can transmit information to the receiver Bob whereas the local measurement of the supervisor Cliff serves as quantum erasure. Cliff can control the entanglement of the quantum channel between Alice and Bob and the average amount of information transmitted from Alice to Bob via a local measurement only on his qubit. It was experimentally demonstrated by Zhang and Jing for continuous variables [24, 25]. Recently, Huang et al. generalized the controlled dense coding protocol of the three-particle GHZ quantum channel to the case of a $(N+2)$-particle GHZ quantum channel where N senders were considered [26].

In this letter, two methods are shown to realize controlled dense coding with a five-qubit cluster state [27]. One of the strategies is that Alice first concentrates the entanglement of the channel between she and Bob, and then performs dense coding. The second one is that Alice doesn't concentrate the channel, she directly applies one of the four unitary operators $\{I, \sigma_X, i\sigma_Y, \sigma_Z\}$ on her qubit and then sends it to Bob, then Bob can obtain 2 bits of information with a certain probability via performing a projective measurement and a generalized measurement described by positive-operator-valued measure (POVM) elements on his two qubit states [28]. Compared with the schemes with GHZ quantum channel, our scheme is more robust against decoherence because a five-qubit cluster state instead of a GHZ state is used as the quantum channel.

2 Controlled Dense Coding via Entanglement Concentration

Let us assume that the quantum channel between the sender Alice, the receiver Bob and the supervisor Cliff, is a one-dimensional five-qubit cluster state which is given by

$$|\Psi\rangle_{12345} = \frac{1}{2}(|00000\rangle + |00111\rangle + |11010\rangle + |11101\rangle)_{12345}, \tag{1}$$

where qubit 1 is hold by Alice, qubit 2 by Bob and qubits 3, 4 and 5 by Cliff, respectively. In order to control the quantum channel between Alice and Bob and the amount of information transmitted from Alice to Bob, Cliff performs a von Neumann measurement on his qubits 3, 4 and 5 under the bases

$$\begin{aligned}
|\phi_1\rangle_{345} &= \cos\theta|000\rangle_{345} + \cos\theta|111\rangle_{345} + \sin\theta|101\rangle_{345} + \sin\theta|010\rangle_{345},\\
|\phi_2\rangle_{345} &= \sin\theta|000\rangle_{345} + \sin\theta|111\rangle_{345} - \cos\theta|101\rangle_{345} - \cos\theta|010\rangle_{345},\\
|\phi_3\rangle_{345} &= \cos\theta|000\rangle_{345} - \cos\theta|111\rangle_{345} + \sin\theta|101\rangle_{345} - \sin\theta|010\rangle_{345},\\
|\phi_4\rangle_{345} &= \sin\theta|000\rangle_{345} - \sin\theta|111\rangle_{345} + \cos\theta|101\rangle_{345} - \cos\theta|010\rangle_{345},\\
|\phi_5\rangle_{345} &= \cos\theta|001\rangle_{345} + \cos\theta|110\rangle_{345} + \sin\theta|100\rangle_{345} + \sin\theta|011\rangle_{345},\\
|\phi_6\rangle_{345} &= \sin\theta|001\rangle_{345} + \sin\theta|110\rangle_{345} - \cos\theta|100\rangle_{345} - \cos\theta|011\rangle_{345},
\end{aligned} \tag{2}$$

$$|\phi_7\rangle_{345} = \cos\theta|001\rangle_{345} - \cos\theta|110\rangle_{345} + \sin\theta|100\rangle_{345} - \sin\theta|011\rangle_{345},$$
$$|\phi_8\rangle_{345} = \sin\theta|001\rangle_{345} - \sin\theta|110\rangle_{345} + \cos\theta|100\rangle_{345} - \cos\theta|011\rangle_{345},$$

(where θ is a measured angle with the region $0 < \theta \leq \pi/4$) and informs his measurement result to Alice and Bob through a classical channel. It is noted that the five-qubit cluster state can be rewritten as

$$|\Psi\rangle_{12345} = \frac{1}{\sqrt{2}}(|\xi\rangle_{12} \otimes |\phi_1\rangle_{345} + |\zeta\rangle_{12} \otimes |\phi_2\rangle_{345}), \tag{3}$$

in the above bases, with

$$\begin{aligned} |\xi\rangle_{12} &= \cos\theta|00\rangle_{12} + \sin\theta|11\rangle_{12}, \\ |\zeta\rangle_{12} &= \sin\theta|00\rangle_{12} - \cos\theta|11\rangle_{12}. \end{aligned} \tag{4}$$

Obviously, Cliff's measurement gives two results $|\phi_1\rangle_{345}$ and $|\phi_2\rangle_{345}$ with the same probability $\frac{1}{2}$. Corresponding to the measurement result $|\phi_1\rangle_{345}$ or $|\phi_2\rangle_{345}$, the state of qubits 1 and 2 collapses to $|\xi\rangle_{12}$ or $|\zeta\rangle_{12}$, respectively. Generally, the states $|\xi\rangle_{12}$ and $|\zeta\rangle_{12}$ are not maximally entangled, and their entanglement described by concurrence [29, 30] can be read as

$$C = 2\cos\theta\sin\theta, \tag{5}$$

so the success probability of transmitting 2 bits of information with them (i.e. the success probability of dense coding) is less than 1.

Now we analyze first the case in which Cliff's measurement gives $|\phi_1\rangle_{345}$ and the state of qubits 1 and 2 collapses to $|\xi\rangle_{12}$. After receiving the measurement result, Alice introduces an auxiliary qubit with original state $|0\rangle_{aux}$ and performs a unitary transformation

$$U_1 = \begin{pmatrix} \tan\theta & 0 & \sqrt{1-\tan^2\theta} & 0 \\ 0 & 1 & 0 & 0 \\ \sqrt{1-\tan^2\theta} & 0 & -\tan\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{6}$$

on the auxiliary qubit and her qubit 1 under the basis $\{|0\rangle_{aux}|0\rangle_1, |0\rangle_{aux}|1\rangle_1, |1\rangle_{aux}|0\rangle_1, |1\rangle_{aux}|1\rangle_1\}$. The collective unitary transformation $U_1 \otimes I_2$ transforms the state $|0\rangle_{aux} \otimes |\xi\rangle_{12}$ to

$$\begin{aligned} |\xi\rangle_{aux12} &= \sqrt{2}\sin\theta|0\rangle_{aux} \otimes \left[\frac{1}{\sqrt{2}}(|00\rangle_{12} + |11\rangle_{12})\right] \\ &\quad + \cos\theta\sqrt{1-\tan^2\theta}|1\rangle_{aux} \otimes |00\rangle_{12}. \end{aligned} \tag{7}$$

Then Alice performs a von Neumann measurement on the auxiliary qubit under the bases $\{|0\rangle_{aux}, |1\rangle_{aux}\}$ and informs Bob the result. If she obtains $|0\rangle_{aux}$, qubits 1 and 2 are maximally entangled. After performing one of the four unitary transformations $\{I, \sigma_X, i\sigma_Y, \sigma_Z\}$ on qubit 1 (where I is a identity matrix and σ_X, σ_Y and σ_Z are Pauli matrices), Alice sends it to Bob. Then Bob knows he has two qubits in one of the four Bell states resulted from Alice's transformation. By performing a Bell-basis measurement, Bob can discriminate Alice's unitary transformation on qubit 1, so 2 bits of information are transmitted. However, if Alice obtains $|1\rangle_{aux}$, qubits 1 and 2 are unentangled. Bob can extract only 1 bit of information with it. So, in this case, on average

$$I_1^{|\xi\rangle_{12}} = 2 \times 2\sin^2\theta + 1 \times (\cos^2\theta - \sin^2\theta) = \cos^2\theta + 3\sin^2\theta \tag{8}$$

bits of information are transmitted from Alice to Bob.

If Cliff's measurement result is $|\phi_2\rangle_{345}$, Alice's unitary transformation on the auxiliary qubit and qubit 1 under the basis $\{|0\rangle_{aux}|0\rangle_1, |0\rangle_{aux}|1\rangle_1, |1\rangle_{aux}|0\rangle_1, |1\rangle_{aux}|1\rangle_1\}$ is given by

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\tan\theta & 0 & \sqrt{1-\tan^2\theta} \\ 0 & 0 & 1 & 0 \\ 0 & \sqrt{1-\tan^2\theta} & 0 & \tan\theta \end{pmatrix}. \tag{9}$$

The collective unitary transformation $U_2 \otimes I_2$ transforms the state $|0\rangle_{aux} \otimes |\zeta\rangle_{12}$ to

$$|\zeta\rangle_{aux12} = \sqrt{2}\sin\theta|0\rangle_{aux} \otimes \left[\frac{1}{\sqrt{2}}(|00\rangle_{12} + |11\rangle_{12})\right] - \cos\theta\sqrt{1-\tan^2\theta}|1\rangle_{aux} \otimes |00\rangle_{12}. \tag{10}$$

Then Alice performs a von Neumann measurement on the auxiliary qubit under the bases $\{|0\rangle_{aux}, |1\rangle_{aux}\}$. If Alice gets the result $|1\rangle_{aux}$, the state of qubits 1, 2 is unentangled, and 1 bit of information can be transmitted. If she gets $|0\rangle_{aux}$, the state of qubits 1 and 2 is maximally entangled, and 2 bits of information can be transmitted. So in the case, Alice can transmit

$$I_1^{|\zeta\rangle_{12}} = 2 \times 2\sin^2\theta + 1 \times (\cos^2\theta - \sin^2\theta) = \cos^2\theta + 3\sin^2\theta \tag{11}$$

bits of information on average.

Synthesizing the two measurement cases, the average amount of information transmitted from Alice to Bob adds up to

$$I_1 = \frac{1}{2} \times I_1^{|\xi\rangle_{12}} + \frac{1}{2} \times I_1^{|\zeta\rangle_{12}} = 1 + 2\sin^2\theta \tag{12}$$

bits.

From (8) and (11) we can see that the entanglement of the channel between Alice and Bob and the average amount of information transmitted from Alice to Bob does not depend on Cliff's measurement result $|\phi_1\rangle_{345}$ or $|\phi_2\rangle_{345}$, but on the measurement angle θ only, that is to say, by adjusting the value of θ, Cliff can control the transmission from Alice to Bob, of course, which is also the control to the entanglement of the channel between Alice and Bob.

3 Controlled Dense Coding with Generalized Measurement

In this section, we only consider the case of Cliff's measurement is $|\phi_1\rangle_{345}$, and the other case can be deduced in a similar fashion. After receiving the measurement result, Alice doesn't concentrate the channel, she directly performs any one of the four unitary operators $\{I, \sigma_X, i\sigma_Y, \sigma_Z\}$ on her qubit 1. Depending on the applied unitary transformation, the shared state $|\xi\rangle_{12}$ undergoes one of the following transformations:

$$\begin{aligned}
&(I \otimes I)|\xi\rangle_{12} = \cos\theta|00\rangle_{12} + \sin\theta|11\rangle_{12} = |\mu_1\rangle_{12},\\
&(\sigma_X \otimes I)|\xi\rangle_{12} = \cos\theta|10\rangle_{12} + \sin\theta|01\rangle_{12} = |\mu_2\rangle_{12},\\
&(i\sigma_Y \otimes I)|\xi\rangle_{12} = -\cos\theta|10\rangle_{12} + \sin\theta|01\rangle_{12} = |\mu_3\rangle_{12},\\
&(\sigma_Z \otimes I)|\xi\rangle_{12} = \cos\theta|00\rangle_{12} - \sin\theta|11\rangle_{12} = |\mu_4\rangle_{12}.
\end{aligned} \tag{13}$$

After this operation, Alice sends qubit 1 to Bob, and now Bob has at his disposal two qubits which could be in any one of the four possible states $\{|\mu_1\rangle_{12}, |\mu_2\rangle_{12}, |\mu_3\rangle_{12}, |\mu_4\rangle_{12}\}$. If Bob is able to distinguish all the four nonorthogonal states conclusively, he can extract two classical bits of information. However, we can find that the above four states are not mutually orthogonal. According to quantum theory, these four non-orthogonal states cannot be distinguished conclusively. But it is well known that a set of non-orthogonal states which are linearly independent can be distinguished with some probability of success [31–33]. Fortunately, it is easy to find that the above set $\{|\mu_1\rangle_{12}, |\mu_2\rangle_{12}, |\mu_3\rangle_{12}, |\mu_4\rangle_{12}\}$ is actually linearly independent. Therefore, Bob can conclusively distinguish these states with some probabilities of success.

In order to distinguish the above set $\{|\mu_1\rangle_{12}, |\mu_2\rangle_{12}, |\mu_3\rangle_{12}, |\mu_4\rangle_{12}\}$, first Bob performs a projection onto the subspaces spanned by the basis states $\{|00\rangle, |11\rangle\}$ and $\{|01\rangle, |10\rangle\}$ with corresponding projective operators $P_1 = |00\rangle\langle 00| + |11\rangle\langle 11|$ and $P_2 = |01\rangle\langle 01| + |10\rangle\langle 10|$ respectively. It is obvious that P_1 and P_2 are mutually orthogonal, so Bob can discriminate the two subsets of Alice's operators: $\{I, \sigma_Z\}$ or $\{\sigma_X, i\sigma_Y\}$. If Bob obtains P_1, then he knows that the state will be either $|\mu_1\rangle_{12}$ or $|\mu_4\rangle_{12}$. Similarly, if he obtains P_2, the state will be either $|\mu_2\rangle_{12}$ or $|\mu_3\rangle_{12}$. After this projective measurement he gets 1 bit of information [34]. Now we suppose that Bob obtains P_1, then he performs a generalized measurement on his two qubit states. In the case, the corresponding positive operator valued measure (POVM) elements in the subspace $\{|00\rangle, |11\rangle\}$ are

$$\begin{aligned}
M_1 &= \frac{1}{2}\begin{pmatrix} \sin^2\theta & \frac{1}{2}\sin 2\theta \\ \frac{1}{2}\sin 2\theta & \cos^2\theta \end{pmatrix},\\
M_2 &= \frac{1}{2}\begin{pmatrix} \sin^2\theta & -\frac{1}{2}\sin 2\theta \\ -\frac{1}{2}\sin 2\theta & \cos^2\theta \end{pmatrix},\\
M_3 &= \begin{pmatrix} 1-\tan^2\theta & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{14}$$

It is easy to check that the condition $M_1 + M_2 + M_3 = I$ is satisfied. The generalized measurement has three outcomes, which is independent of the state of the measured system. Therefore, the generalized measurement provides the most general physically realized measurement in quantum mechanics.

If Bob gets M_1 then he can sure that the state of qubits 1 and 2 is $|\mu_1\rangle_{12}$, if he gets M_2 then the state is $|\mu_4\rangle_{12}$, and if he gets M_3 the state is completely indecisive, he can get nothing about the two states. The success probability of distinguishing $|\mu_1\rangle_{12}$ and $|\mu_4\rangle_{12}$ is $2\sin\theta^2$, which is also the probability that Bob obtains another bit of information. Similar procedure can be applied to the case of M_2, it is easy to find that the relevant POVM elements and the success probability are the same. Hence the amount of information transmitted from Alice to Bob is

$$I_2^{|\xi\rangle_{12}} = 1 + 2\sin^2\theta \tag{15}$$

bits.

Similarly, if Cliff's measurement result is $|\phi_2\rangle_{345}$, with the same procedure, one can show that the amount of information is also given by the above expression. Synthesizing all the two measurement cases, the average amount of information transmitted from Alice to Bob can be expressed as

$$I = \frac{1}{2} \times I_2^{|\xi\rangle_{12}} + \frac{1}{2} \times I_2^{|\zeta\rangle_{12}} = 1 + 2\sin^2\theta. \qquad (16)$$

Therefore, it is helpful for Cliff to control the average amount of information transmitted from Alice to Bob by adjusting the measured angle. Comparing (12) with (16), we find that the results are same, which means that the two schemes are equivalent for the controlled dense coding and their results are unique.

4 Summary

In summary, two schemes, via entanglement concentration and generalized measurement respectively, of realizing controlled dense coding are investigated by a one-dimensional five-qubit cluster state. It is shown that the entanglement of the channel between Alice and Bob and the average amount of information only depend on Cliff's measured angle θ, which implies that Cliff can control the entanglement of the channel and the average amount of information transmitted from Alice to Bob by adjusting the measured angle θ. It is also shown that the results for the average amounts of information are unique from the different two schemes.

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